

Wilson Loops and Topological Phases in Closed String Theory

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Published Online: September 22, 2006

Using covariant phase space formulations for the natural topological invariants associated with the world-surface in closed string theory, we find that certain Wilson loops defined on the world-surface and that preserve topological invariance, correspond to wave functionals for the vacuum state with zero energy. The differences and similarities with the 2-dimensional QED proposed by Schwinger early are discussed.

KEY WORDS: string theory; quantization; topological field theory.

PACS Numbers : 81T30, 81T45

1. INTRODUCTION

At present one of the fundamental problems in string theory is the understanding of its nonperturbative aspects, intimately related with the search for the underlying geometry of the theory, which will allow us, for example, to determine the corresponding true vacuum. In this sense, the development of topological field theory constitutes an important step in such a direction, specifically in the construction of the called *unbroken phase*, where the relevant symmetries of the theory such as general covariance are unbroken (Kaku, 1999).

Although it is undeniable the fundamental role that the topological invariants of the theory play in this context, it is strange that we have not yet a more profound understanding of the underlying geometry of those topological invariants that emerge in a natural way in string theory, i.e., the Euler characteristic and the second Chern number of the worldsheet. However, in (Cartas-Fuentevilla, 2004) we have demonstrated that the underlying geometrical structure in a covariant phase space formulation for such topological string actions mimics that of a two-dimensional U(1)-gauge theory, whose geometrical meaning is perhaps the most understood. Therefore, in the present work, we attempt to exploit the results in

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(Cartas-Fuentevilla, 2004) and to gain insight into the geometrical structure of these topological invariants and their intimate relations with the vacuum state of the theory.

The main result will be that in a second quantization scheme for these topological string actions, the wave functionals for the vacuum state with zero energy correspond in a natural way to the Wilson loops of certain connections characterizing the two-dimensional world-surface along the spatial configuration of the closed string.

This work is organized as follows. In the next section, we give an outline on the two-dimensional world-surface that will be useful in the developed of the work. In Section 3, we describe briefly the covariant phase space formulation given in (Cartas-Fuentevilla, 2004) for the Euler characteristic. In Section 4, the quantization and the Wilson loops as wave functionals for the ground state are considered. In Section 5, the constraints and symmetries are discussed, and subsequently in Section 6 the covariance and topological invariance of the Wilson loops are analyzed. The second Chern number is considered in Section 7, and we finish in Section 8 with some concluding remarks.

2. PRELIMINARIES OF THE EMBEDDING 2-SURFACE

It is convenient to give a survey on some basic ideas on the embedding two-surface that will be useful in the present treatment.

Considering the expression for the internal curvature tensor $R_{\mu\nu\alpha\beta}$ of an imbedding given by Carter (1992) in terms of an internal gauge connection ρ , we can find that, for a two-dimensional embedding

$$R_{\mu\nu} = R_{\mu\sigma\nu}{}^\sigma = \frac{1}{2}n_{\nu\alpha}\mathcal{E}^{\alpha\lambda}(\bar{\nabla}_\mu\rho_\lambda - n_\mu^\pi\bar{\nabla}_\lambda\rho_\pi), \tag{1}$$

where $n^{\mu\nu}$ is the fundamental tensor of the imbedding two-surface, which is characterized by the antisymmetric unit tangent element tensor

$$\mathcal{E}^{\mu\nu} = t^\mu\vartheta^\nu - t^\nu\vartheta^\mu, \tag{2}$$

t^μ being a timelike unit vector and ϑ^μ a spacelike unit vector, both tangential to the world-surface. Furthermore, $\rho_\sigma = \mathcal{E}^\nu{}_\mu\rho_\sigma{}^\mu{}_\nu$, where $\rho_\sigma{}^\mu{}_\nu$ corresponds to the background space-time components of the internal frame components of the natural gauge connection for the group of two-dimensional internal frame rotations. Furthermore, $\bar{\nabla}_\mu \equiv n_\mu^\alpha\nabla_\alpha$, where ∇_α is the usual Riemannian covariant differentiation operator associated with the background metric $g_{\mu\nu}$.

From. (1), we find that

$$R_{00} = -\frac{1}{2}\mathcal{E}^{0i}(\bar{\nabla}_0\rho_i - \bar{\nabla}_i\rho_0), \tag{3}$$

where we have considered that $n_{00} = 1, n_{0i} = 0$. Additionally, in (Carter, 1992) it is shown that for a two-dimensional world-surface

$$R = \xi^{\mu\nu} \bar{\nabla}_\mu \rho_\nu = \mathcal{E}^{0i} (\bar{\nabla}_0 \rho_i - \bar{\nabla}_i \rho_0) = t^0 \vartheta^i (\bar{\nabla}_0 \rho_i - \bar{\nabla}_i \rho_0), \tag{4}$$

where (2) has been used. Equations (3), and (4) will be useful below in the construction of the second quantization for the Euler characteristic χ of the two-dimensional world-surface S embedded in an arbitrary background space-time, which can be described in terms of the inner curvature scalar R as (Carter, 1992)

$$\chi = (2 - 2g) = \sigma_1 \int_S \sqrt{-\gamma} R dS, \tag{5}$$

where g is the number of handles of the world surface, and γ the determinant of the embedded surface metric. In (Cartas-Fuentevilla, 2004) it is shown that

$$\delta \chi = \sigma_1 \int_S \sqrt{-\gamma} \left(\frac{1}{2} R n^{\mu\nu} - R^{\mu\nu} \right) \delta g_{\mu\nu} dS, \tag{6}$$

modulo a total divergence; hence the energy-momentum tensor vanishes

$$T^{\mu\nu} = \frac{1}{2} R n^{\mu\nu} - R^{\mu\nu} = 0, \tag{7}$$

as expected for a topological invariant, since

$$\frac{1}{2} R n^{\mu\nu} - R^{\mu\nu} = 0, \tag{8}$$

identically for a two-dimensional surface (Carter, 1992). Although it is evident that at classical level a topological invariant is physically trivial, we shall see that at quantum level the things can be very different.

3. THE PHASE SPACE FORMULATION FOR THE EULER CHARACTERISTIC

In (Cartas-Fuentevilla, 2004) it is shown that the nontrivial phase space formulation for χ is given by a covariant and gauge-invariant symplectic structure

$$\omega = \int_\Sigma \delta(\sqrt{-\gamma} \mathcal{E}^{\mu\nu}) \delta \rho_\nu d\bar{\Sigma}_\mu, \tag{9}$$

the constraint

$$\bar{\nabla}_\mu \mathcal{E}^{\mu\nu} = 0, \tag{10}$$

and the *Bianchi identity*

$$R \mathcal{E}_{\kappa\lambda} = 2n_{[\lambda}{}^\sigma \bar{\nabla}_{\kappa]} \rho_\sigma, \tag{11}$$

which gives the two-form $R\mathcal{E}$ as the *exterior derivative* of the one-form ρ (Carter, 1999)

Equations (9)–(11) mimic in its mathematical structure and symmetry properties the phase space formulation of an Abelian gauge theory (Cartas-Fuentevilla, 2004). However, there exists an important difference since, unlike the conventional (3+1)-dimensional $U(1)$ -gauge theory, (9)–(11) represents a (1+1)-dimensional $U(1)$ -gauge theory *embedded* in an arbitrary ambient space-time. In relation to the dimensionality, (9)–(11) would be closer to the (1+1)-dimensional QED (without massless fermions) proposed by Schwinger early (Schwinger, 1962) and whose first Hamiltonian analysis and solutions were given for example in (Lowenstein and Swieca, 1971; Manton, 1985). The present analysis of the canonical formulation for χ will have some parallelism with (Lowenstein and Swieca, 1971; Manton, 1985) although, of course, also its particularities.

For convenience we shall make a decomposition space+time of the covariant phase space formulation of χ given by (9)–(11). The covariance will be recuperated at the end.

The choice of ι^μ as a timelike vector, and ϑ^μ as a spacelike one, induces naturally a (1+1)-decomposition on the covariant description of the formulation (9)–(11); such a decomposition was already used in the expressions (3), and (4).

In this manner, considering that in (9) $d\bar{\Sigma}_\mu$ is normal to the Cauchy space-like surface Σ , we can choose $d\bar{\Sigma}_\mu = \iota_\mu d\Sigma$, and ω takes the noncovariant form

$$\omega = \int_\Sigma \delta(\sqrt{-\gamma}\mathcal{E}^{0i})\delta\rho_i d\Sigma, \tag{12}$$

which allows us to use a *temporal gauge*

$$\rho_0 = 0, \tag{13}$$

in order to simplify our calculations. Although Σ is strictly a Cauchy hypersurface, the integral on Σ can be reduced actually to an integral on the one-dimensional spatial configuration of the closed string, as we shall do below.

Finally, (10) implies that

$$\bar{\nabla}_i \mathcal{E}^{i0} = 0, \tag{14}$$

in this noncovariant description of the phase space. Equation (14) would represent “the Gauss law constraint,” the analogue of that of the conventional gauge theory.

4. SECOND QUANTIZATION AND WILSON LOOP

Equation (12) shows explicitly that the canonically conjugate phase space variables are $(\sqrt{-\gamma}\iota^0)\mathcal{E}^{0i}$, and the spatial connection ρ_i . In this manner, in order

to make the corresponding quantization we have the correspondence

$$\begin{aligned} \rho_i &\rightarrow \rho_i, \\ \mathcal{E}^{0i} &\rightarrow (l^0) i \frac{\delta}{\delta \rho_i}. \end{aligned} \tag{15}$$

For constructing the quantum Hamiltonian (H_Q) we can use the classical expression for T^{00} given by (3), (4), and (8),

$$T^{00} = \frac{1}{2} [\mathcal{E}^{0i} \bar{\nabla}_0 \rho_i - l^0 \vartheta^i \bar{\nabla}_0 \rho_i], \tag{16}$$

and then,

$$H_Q = \int_{\Sigma} F_i \left(i \frac{\delta}{\delta \rho_i} - \vartheta_i \right) d\Sigma, \tag{17}$$

where $F_i = l^0 \bar{\nabla}_0 \rho_i$; since $[F_i, \frac{\delta}{\delta \rho_i}] = 0$, we have not ordering ambiguity in the Hamiltonian (17).

Furthermore, the ground state wave functionals ψ , a representation of the vacuum state of the theory, satisfy $H_Q \psi = E \psi$, and then for a state of zero energy ψ satisfies

$$\left(i \frac{\delta}{\delta \rho_i} - \vartheta_i \right) \psi = 0, \tag{18}$$

for which the unique solution will be given evidently by

$$\psi(\rho) = A e^{-i \int_{\Sigma} \vartheta_i \rho_i d\Sigma}, \tag{19}$$

where A is a constant parameter. Considering that ϑ_i is a space-like vector tangent to the world-surface, then ϑ_i goes tangentially along the closed string loop, and hence

$$\psi(\rho) = A e^{-i \oint \rho_i d\vartheta_i}. \tag{20}$$

In this manner, our ground state wave-function corresponds, in a natural way, to the Wilson loop for the connection ρ_i . Note that ρ_i is of support confined just on the closed string loop.

5. CONSTRAINTS AND SYMMETRIES

In (Cartas-Fuentevilla, 2004) it is proved that the action (5), and its covariant phase space formulation given in (9)–(11) are invariant under the *gauge transformation* of the connection ρ_{μ} ,

$$\rho_{\nu} \rightarrow \rho_{\nu} + \bar{\nabla}_{\nu} \phi, \tag{21}$$

where ϕ is an arbitrary scalar field. It is in this sense that χ and its covariant canonical formulation mimic the symmetry properties of a $U(1)$ -gauge theory (Cartas-Fuentevilla, 2004).

It is easy to see that under the *spatial restriction* of (21)

$$\rho_i \rightarrow \rho_i + \bar{\nabla}_i \phi, \tag{22}$$

the wave-function ψ (20) is gauge invariant because the string loop ϑ is closed. In this sense ψ represents a physical state for the theory.

Furthermore, the ‘‘Gauss law’’ (14) must be imposed as a constraint on the quantum state ψ ,

$$\bar{\nabla}_i \frac{\delta}{\delta \rho_i} \psi = 0, \tag{23}$$

which implies that ψ must be gauge invariant in the sense of (22), such as in the conventional quantum gauge theory; thus, our wave-function ψ solves automatically the constraints of the theory. In this manner, the ‘‘Gauss law’’ (14) is the generator of the gauge symmetry at quantum level.

At classical level \mathcal{E}^{0i} is essentially ϑ^i , and in the present canonical analysis \mathcal{E}^{0i} is the variable conjugate to ρ_i , and hence we have the primary constraints on the physical states

$$\left(i \frac{\delta}{\delta \rho_i} - \vartheta_i \right) \psi = 0, \tag{24}$$

which corresponds exactly to the equation (18) for the wave-function (20); thus $\psi(\rho)$ solves naturally the primary constraints of the theory. In this sense, the quantum Hamiltonian (17) is a pure combination of the primary constraints.

6. COVARIANCE AND TOPOLOGICAL INVARIANCE OF $\psi(\rho)$

The decomposition space-time used previously is not strictly necessary, and we have employed it only for computational convenience and for making contact with the conventional noncovariant description of quantum gauge theory. However, we can get back the covariance directly on the Wilson loops given in (19). It is evident that $\psi(\rho)$ can be written as

$$\psi(\rho) = A e^{-i \int_{\Sigma} \mathcal{E}^{\mu\nu} \rho_{\mu} d\bar{\Sigma}_{\nu}}, \tag{25}$$

which is manifestly covariant. Note that now (25) is invariant under the gauge transformation (21) since, considering (10)

$$\int_{\Sigma} \mathcal{E}^{\mu\nu} (\rho_{\mu} + \bar{\nabla}_{\mu} \phi) d\Sigma_{\nu} = \int_{\Sigma} \mathcal{E}^{\mu\nu} \rho_{\mu} d\Sigma_{\nu} + \int_{\Sigma} \bar{\nabla}_{\nu} (\mathcal{E}^{\mu\nu} \phi) d\Sigma_{\nu}, \tag{26}$$

where the last integral can be reduced to

$$\int_{\Sigma} \bar{\nabla}_\nu(\mathcal{E}^{\mu\nu} \phi) d\Sigma_\nu = \int_{\partial\Sigma} \mathcal{E}^{\mu\nu} \phi d\Sigma_{\mu\nu}, \tag{27}$$

which vanishes (on $\partial\Sigma$) for fields with compact support in the *spatial directions*.

Furthermore, the covariant expression (25) shows manifestly that the Wilson loop $\psi(\rho)$ does not depend on any background metric structure, and is topological in character. In this manner, the Wilson loop $\psi(\rho)$ describes a topological phase with unbroken diffeomorphism invariance in string theory.

**7. ANOTHER TOPOLOGICAL INVARIANT:
THE SECOND CHERN NUMBER**

For a string in ordinary four-dimensional space-time, we have another topological invariant for the world-surface, the second Chern number of the normal bundle (Carter, 1992)

$$\chi' = \sigma_2 \int_S \sqrt{-\gamma} \Omega dS, \tag{28}$$

where S is the entire imbedding two-surface and σ_2 a parameter; furthermore,

$$\begin{aligned} \Omega &= \bar{\nabla}_\mu(\mathcal{E}^{\mu\nu} \omega_\nu), \\ \omega_\nu &= \frac{1}{2} \omega_\nu^{\mu\lambda} \epsilon_{\lambda\mu\rho\sigma} \mathcal{E}^{\rho\sigma}, \end{aligned}$$

where $\omega_\nu^{\mu\lambda}$ corresponds to the external frame rotation (pseudo-)tensor, and $\epsilon_{\lambda\mu\rho\sigma}$ the antisymmetric background measure tensor. ω_ν is the outer analogue of ρ_μ , the inner gauge connection considered above. Geometrically χ' is related with the number of self-intersections of the two-surface.

In this case, the covariant phase space formulation is given by the symplectic structure

$$\omega' = \int_{\Sigma} \delta(\sqrt{-\gamma} \mathcal{E}^{\mu\nu}) \delta\omega_\nu d\bar{\Sigma}_\mu, \tag{29}$$

the constraint (10), and the Bianchi identity

$$\Omega \mathcal{E}_{\mu\nu} = 2n_{[\nu}{}^\sigma \bar{\nabla}_{\mu]} \omega_\sigma. \tag{30}$$

The corresponding *gauge transformation* of the connection ω_μ is given similarly by (see (21)),

$$\omega_\mu \rightarrow \omega_\mu + \bar{\nabla}_\mu \phi, \tag{31}$$

under which, the action (28) and the two-form (30) are strict invariants; ω' in (29) turns out to be invariant modulo a total divergence. The corresponding Wilson

loop for the outer connection ω_μ ,

$$\psi'(\omega_\mu) = B e^{-i \int_\Sigma \mathcal{E}^{\mu\nu} \omega_\mu d\bar{\Sigma}_\nu}, \quad (32)$$

is also invariant under (31) in an entirely similar way to $\psi(\rho)$ in Section 6. ψ' is, of course, a topological invariant too, and represents hence a topological phase for the theory.

8. CONCLUDING REMARKS

In this manner, we have started from topological string actions and finished with Wilson loops as representations of the vacuum state, that also are topological invariants in character. Hence, as a direct consequence of such a topological invariance, the *general covariance* of the theory is preserved as an unbroken symmetry.

On the other hand, and independently on the results obtained here, it is well known that the Wilson loops represent a possible scheme for the quantum theories of connections such as Yang-Mills and gravity, where all gauge invariant information in a connection is contained in the Wilson loops. Hence, we can finish with an open question that may be the subject of forthcoming works: may the Wilson loops founded in the present work constitute the fundamental blocks for a reformulation of string theory in the search for its unbroken phase?

ACKNOWLEDGMENTS

The author acknowledges the support from the Sistema Nacional de Investigadores (México).

REFERENCES

- Cartas-Fuentevilla, R. (2004). *Fluctuating topological invariants in string theory as an Abelian gauge theory*, to be published, International Journal of Modern Physics, math-ph/0404011.
- Carter, B. (1992). *Journal of Geometry and Physics* **8**, 53.
- Kaku, M. (1999). *Strings, conformal fields, and M-theory*, Springer, New York, Chapter 12.
- Lowenstein, J. H. and Swieca, J. A. (1971). *Annals of Physics* **68**, 172; Manton, N. S. (1985). *Annals of Physics* **159**, 220.
- Schwinger, J. (1962). *Physical Review* **128**, 2425